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Comparison of Optimal Control and Differential Game Intercept Missile Guidance Laws

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Air-to-air missile guidance laws are derived using optimal control and differential game theory with final miss distance as the optimization criterion. A perfect target airframe/autopilot response is assumed, while both perfect and first-order missile responses are considered. With a first-order missile response the target is always able to force a nonzero final miss distance in the differential game formulation. For all other formulations considered there are states from which the missile can force zero terminal miss. In these cases, an auxiliary performance index (e.g., control energy) can be used to specify unique controls. Two simulation scenarios were used to evaluate the guidance laws: one with missile launch near the inner launch boundary and the other near the outer launch boundary. The differential game guidance laws are less sensitive to errors in estimates of current target acceleration than the optimal control laws. The laws based on a perfect missile response performed better for the outer launch boundary scenario; whereas for the inner launch boundary scenario the laws based on a first-order missile response achieved smaller miss distances.

Nomenclature

b_m, b_t	= weights on missile and target control terms in differential game quadratic performance index
F_1, F_2, F_3, F_4	= functions defined by Eq. (37)
J	= performance index
M_p	= predicted final miss
$\tilde{M}_p$	= defined by Eq. (43)
$M_{p\mathrm{tol}}$	= tolerance on predicted final miss
t	= time
t^*	= optimal target maneuver time in differential game formulation
t_0	= initial time
t_f	= final time
u	= missile control normal to reference line-of-sight (LOS)
u^*	= optimal missile control
$u_{\max}$	= bound on missile control
v	= target acceleration control normal to reference LOS
v^*	= optimal target control in differential game formulations
$\bar{v}$	= projected target acceleration profile in optimal control formulations
$\hat{v}$	= estimate of current target acceleration
$v_{\max}$	= bound on target control
V_M	= missile speed
V_T	= target speed
y_m, y_T	= missile and target position coordinates normal to reference LOS
y_1, y_2	= relative position and velocity coordinates normal to reference LOS
y_3	= actual missile acceleration normal to reference LOS
$y_{3\mathrm{tol}}$	= tolerance on y_3
λ_m, λ_3	= costate variables
τ	= missile airframe/autopilot time constant

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I. Introduction

THE basic difference in philosophy between missile guidance laws based on optimal control theory and those based on differential game theory is in the assumptions made by the guidance laws on the target's future trajectory and maneuvering capabilities. Optimal control theory assumes that the target's future maneuver strategy is completely defined, either in open-loop or closed-loop form. The feedback nature of missile guidance laws allows the missile to correct for inaccurate predictions of target maneuvers. For example, proportional navigation is an optimal guidance law for a particular set of intercept dynamics, performance index, and terminal constraints with the assumption of straight line flight (i.e., no maneuvering) by the target in the future.

In contrast, the differential game approach makes no assumption on future target maneuvers, but instead takes into consideration the target's maneuver capabilities. The guidance law then guides the missile so as to minimize the potential effects of the target's intelligent use of his maneuver capabilities.

Optimal control theory has been used to derive a variety of deterministic guidance laws for intercept missiles.[1-9] These laws are all based on the application of linear-quadratic optimal control theory to a linear constant coefficient missile model with various assumptions on availability of target acceleration information, enforcement of zero final miss, and the model used for the airframe/autopilot response of the missile. In all cases the resulting optimal guidance law is a modified form of proportional navigation.

In contrast, the application of zero sum perfect information differential game theory to the derivation of intercept missile guidance laws has been less extensive.[9-13] A complicated numerical technique for the generation of near optimal feedback solutions to nonlinear zero sum differential games was proposed in Ref. 10 as a guidance law. Because of its complexity, it currently does not appear to be implementable in real time. Linear-quadratic differential game theory was used in Ref. 9 to derive a form of modified proportional navigation. References 11-13 derived guidance laws based on the minimization of final miss distance using linear constant coefficient missile models.

In these previous works on the derivation of guidance laws using optimal control and differential game theory, no attempt has been made to compare the differential game laws with similar optimal control laws, either from the viewpoint of the basic structure of the laws themselves or the performance of the resulting laws in a realistic simulation. The basic purpose of this paper is to fill this gap, i.e., to compare air-to-air missile guidance laws derived using optimal control and differential game methods with similar missile models, performance indices, and constraints. A linear constant velocity planar model of the intercept problem is used in this analysis. A perfect target airframe/autopilot response is assumed throughout, while both perfect and first-order missile airframe/autopilot responses are considered.

The next section describes the dynamic formulation of the intercept problem. Guidance laws based on perfect and first-order missile airframe/autopilot responses are derived in Secs. III and IV. This is followed by a comparison of these laws using simulation results. A summary of conclusions is then presented in Sec VI.

II. Dynamic Formulation

In this section we consider the dynamic formulation of the planar intercept problem used in the derivation of the guidance laws. First, the formulation based on a perfect missile airframe/autopilot response is discussed, followed by the modifications required with a first-order response. A brief discussion of the basic performance index used in the guidance law derivations concludes this section.

Perfect Airframe/Autopilot Response

Figure 1 shows the planar constant velocity intercept problem. The dynamics of the target and missile will be derived relative to a reference inertial line-of-sight (LOS) direction which will remain fixed if the missile is on a collision course with the target. The displacement of the target normal to this reference is y_T and the displacement of the missile is y_m. It is convenient to define the position of the target relative to the missile by

$$y_1 = y_T - y_m \tag{1}$$

Differentiating Eq. (1) twice we obtain the state equations

$$\dot{y}_1 = y_2 \qquad \dot{y}_2 = v - u \tag{2}$$

where v is the target acceleration and u is the commanded (and actual) missile acceleration component normal to the reference LOS.

In this formulation we will assume that the time-to-go $(t_f - t)$ is known or estimated. In the optimal control formulations the target acceleration component v is assumed to be a known or estimated constant for the remainder of the flight. This latter assumption can be removed at a later time. In the differential game formulation v is treated as a control variable.

A simpler formulation results if we introduce the concept of predicted miss M_p defined by

$$M_p = y_1 + y_2 (t_f - t) \tag{3}$$

Note that M_p is the miss distance at t_f if no missile or target control is used from time t to t_f. Differentiating Eq. (3) and using Eqs. (2) to substitute for $\dot{y}_1$ and $\dot{y}_2$, we obtain

$$\dot{M}_p = (v - u)(t_f - t) \tag{4}$$

This single differential equation is equivalent to Eqs. (2).

In general, the missile and target commanded accelerations normal to the reference LOS are constrained by

$$|u| \le u_{\max} \qquad |v| \le v_{\max} \tag{5}$$

These inequalities will be enforced in the derivations of the guidance laws.

First-Order Airframe/Autopilot Response

The state equations for the position and velocity of the target relative to the missile can now be expressed as

$$\dot{y}_1 = y_2 \qquad \dot{y}_2 = v - y_3 \tag{6}$$

where y_3 is the actual missile acceleration normal to the reference LOS. Assuming a first-order airframe/autopilot response, the differential equation for y_3 is

$$\dot{y}_3 = (-y_3 + u)/\tau \tag{7}$$

where τ is the time constant and u is now the commanded missile acceleration component normal to the LOS.

A simpler formulation again results if we replace the variables y_1 and y_2 by the predicted miss M_p defined by Eq. (3) giving

$$\dot{M}_p = (v - y_3)(t_f - t) \tag{8}$$

Equations (7) and (8) describe the missile/target dynamics that will be used to derive the guidance laws based on a first-order airframe/autopilot response for the missile.

Performance Index

The basic function of the guidance system of an air-to-air missile is to steer the missile in such a way as to minimize the final miss distance to the target. In contrast, the target should attempt to maximize this final miss distance. Therefore, the basic performance index for this intercept problem can be specified as

$$J = M_p(t_f)^2/2 \tag{9}$$

In both the optimal control and differential game formulations the guidance law should specify the missile control u that minimizes this quantity. In the differential game formulations, the possibility that the target can choose his control v so as to maximize Eq. (9) is considered.

III. Guidance Laws: Perfect Missile Airframe/Autopilot Response

First the guidance laws will be presented for the optimal control approach, followed by the corresponding laws obtained using differential game theory.

Optimal Control Approach

With the target control v considered as an assumed constant $\bar{v}$ for the remainder of the flight, the application of the necessary conditions for an optimal control gives the following two-point boundary-value problem (TPBVP) with

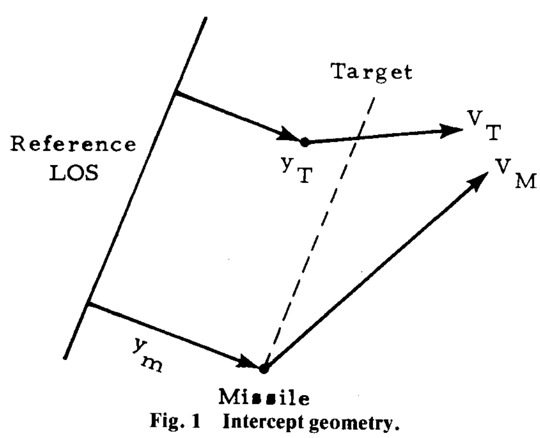

Fig. 1 Intercept geometry.

t_f assumed known:

$$\dot{M}_p = (\bar{v} - u^*)(t_f - t) \qquad M_p(t_0) \text{ known}$$
$$\dot{\lambda}_m = 0 \qquad \lambda_m(t_f) = M_p(t_f)$$
$$u^* = \begin{cases} u_{\max} & \text{if } \lambda_m > 0 \\ \text{nonunique} & \text{if } \lambda_m = 0 \\ -u_{\max} & \text{if } \lambda_m < 0 \end{cases} \quad (10)$$

This TPBVP can be solved to obtain the following feedback control law when u^* is uniquely defined:

$$u^*(t) = u_{\max} \quad \text{if } M_p(t) + (\bar{v} - u_{\max})(t_f - t)^2/2 > 0$$
$$= -u_{\max} \quad \text{if } M_p(t) + (\bar{v} + u_{\max})(t_f - t)^2/2 < 0 \quad (11)$$

No unique solution exists if

$$|M_p(t) + \bar{v}(t_f - t)^2/2| - u_{\max}(t_f - t)^2/2 < 0 \quad (12)$$

i.e., there are an infinite number of control strategies that can reduce $|M_p(t_f)|$ to zero. There are a variety of ways to implement this nonunique optimal control. Probably the simplest is to use $u^* = \pm u_{\max}$ to reduce $M_p(t)$ to zero, and then use $u^* = \bar{v}$ to maintain $M_p(t) = 0$. This can be expressed by the following optimal control, perfect autopilot (OCP) control law.

Control Law OCP 1:

$$u^* = u_{\max}\operatorname{sgn}[M_p(t)] \quad \text{if } |M_p(t)| > M_{ptol}$$
$$= \bar{v} \quad \text{if } |M_p(t)| < M_{ptol} \quad (13)$$

where M_{ptol} is a tolerance on M_p.

A similar approach is to use $u^* = \pm u_{\max}$ to reduce the predicted value of $M_p(t_f)$ to zero, with $u^* = 0$ used thereafter. This results in the control law

Control Law OCP 2:

$$u^* = u_{\max}\operatorname{sgn}\left[M_p(t) + \frac{\bar{v}(t_f - t)^2}{2}\right] \quad \text{if } \left|M_p(t) + \frac{\bar{v}(t_f - t)^2}{2}\right| > M_{ptol}$$
$$= 0 \quad \text{otherwise} \quad (14)$$

When the nonunique control situation occurs, we can enforce $M_p(t_f) = 0$ in the optimal control problem formulation while minimizing another performance index. A common performance index is the control energy defined by

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2 \mathrm{d}t \quad (15)$$

The TPBVP resulting from this formulation is

$$\dot{M}_p = (\bar{v} - u^*) \qquad M_p(t_0) \text{ known}$$
$$\dot{\lambda}_m = 0 \qquad M_p(t_f) = 0$$
$$u^* = \lambda_m(t_f - t) \qquad |u| \le u_{\max} \quad (16)$$

This TPBVP can be solved to generate the following control law.

Control Law OCP 3:

$$u^* = u_{\max} \quad \text{if } u_i \ge u_{\max}$$
$$= u_i \quad \text{if } |u_i| < u_{\max}$$
$$= -u_{\max} \quad \text{if } u_i \le -u_{\max} \quad (17)$$

where

$$u_i = \frac{3[M_p + \bar{v}(t_f - t)^2/2]}{(t_f - t)^2}$$

It should be noted that control law OCP 3 uses $|u^*| = u_{\max}$ as long as

$$|M_p + \bar{v}(t_f - t)^2/2| > |u_{\max}(t_f - t)^2/3| \quad (18)$$

Therefore, the optimal control specified by Eqs. (11) is automatically enforced by this control law.

Differential Game Approach

Now the target is assumed to have the capability of using an optimal control in order to evade the missile. With the performance index given by Eq. (9), the application of the necessary conditions for a saddle point solution gives the TPBVP

$$\dot{M}_p = (v^* - u^*)(t_f - t) \qquad M_p(t_0) \text{ known}$$
$$\dot{\lambda}_m = 0 \qquad \lambda_m(t_f) = M_p(t_f)$$
$$u^* = u_{\max} \text{ and } v^* = v_{\max} \quad \text{if } \lambda_m > 0$$
$$u^*, v^* \text{ nonunique} \quad \text{if } \lambda_m = 0$$
$$u^* = -u_{\max} \text{ and } v^* = -v_{\max} \quad \text{if } \lambda_m < 0 \quad (19)$$

When the optimal controls are uniquely defined ($\lambda_m \neq 0$), the TPBVP can be solved to obtain

$$u^*(t) = u_{\max} \quad \text{if } M_p(t) + (v_{\max} - u_{\max})(t_f - t)^2/2 > 0$$
$$= -u_{\max} \quad \text{if } M_p(t) - (v_{\max} - u_{\max})(t_f - t)^2/2 < 0 \quad (20)$$

If $v_{\max} > u_{\max}$, the target has greater maneuverability and is able to force $|M_p(t_f)| > 0$. However, we are more interested in the case when $u_{\max} > v_{\max}$, i.e., the missile has more maneuverability. In this case, a nonunique solution exists when

$$|M_p(t)| - |(v_{\max} - u_{\max})|(t_f - t)^2/2 < 0 \quad (21)$$

From a gaming viewpoint, the most desirable state of the problem for the missile is $M_p(t) = 0$. By keeping $M_p(t) = 0$, the missile is able to minimize the evasive capabilities of the target. A control law which performs this function for the differential game, perfect autopilot (DGP) problem is

Control Law DGP 1:

$$u^* = u_{\max}\operatorname{sgn}[M_p(t)] \quad \text{if } |M_p(t)| > M_{ptol}$$
$$= \hat{v} \quad \text{if } |M_p(t)| \le M_{ptol} \quad (22)$$

where $\hat{v}$ is the estimate of the current target acceleration. Note that this control law is identical to OCP 1. Since this gaming approach does not project a future target acceleration, no differential game control law equivalent to OCP 2 exists.

When $u_{\max} > v_{\max}$ and the nonunique control situation occurs, the missile is able to force $M_p(t_f) = 0$. Another performance index can now be used to define unique solutions. A typical performance index used in differential games is

$$J=\frac{1}{2}\int_{t_0}^{t_f}(b_m u^2-b_t v^2)\,dt \tag{23}$$

where weighting parameters b_m and b_t will be defined later.

The TPBVP for this new differential game problem is

$$\begin{aligned}
&\dot{M}_p=(v^*-u^*)(t_f-t) && M_p(t_0)\ \text{known}\\
&\dot{\lambda}_m=0 && M_p(t_f)=0\\
&u^*=\lambda_m(t_f-t)/b_m && |u|\le u_{\max}\\
&v^*=\lambda_m(t_f-t)/b_t && |v|\le v_{\max}
\end{aligned} \tag{24}$$

This TPBVP can be solved to obtain the following feedback control for the missile when $|u^*|<u_{\max}$

$$u^*=\frac{3M_p}{(1-b_m/b_t)(t_f-t)^2} \tag{25}$$

To define the ratio b_m/b_t, let us require Eq. (25) to give $|u|=u_{\max}$ when M_p is on one of the boundaries defined by Eqs. (20). This gives

$$(1-b_m/b_t)=3(1-v_{\max}/u_{\max})/2 \tag{26}$$

The resulting expression for u^* is

$$u^*=\frac{2M_p}{(1-v_{\max}/u_{\max})(t_f-t)^2} \tag{27}$$

The control laws given by Eqs. (20) and (27) can be combined into one law as follows.

Control Law DGP 2:

$$\begin{aligned}
u^*&=u_{\max} && \text{if } M_p>(u_{\max}-v_{\max})(t_f-t)^2/2\\
&=u_i && \text{if } |M_p|\le(u_{\max}-v_{\max})(t_f-t)^2/2\\
&=-u_{\max} && \text{if } M_p<-(u_{\max}-v_{\max})(t_f-t)^2/2
\end{aligned} \tag{28}$$

where

$$u_i=\frac{2M_p}{(1-v_{\max}/u_{\max})(t_f-t)^2}$$

This guidance law is identical to the one derived in Ref. 11. Note that the factor multiplying M_p increases as $v_{\max}/u_{\max}$ increases. In implementing this law, it is important that a pessimistic value of $v_{\max}$ be used since too low an assumed value can allow the target to escape to the region defined by Eqs. (20) in which a nonzero miss will occur if the target uses its optimal control.

If $v_{\max}/u_{\max}<1/3$, Eq. (26) requires that $b_t<0$ (assuming $b_m>0$) which would invalidate the differential game analysis based on the performance index given by Eq. (23). This difficulty can be avoided if we arbitrarily define b_m/b_t by

$$(1-b_m/b_t)=(1-v_{\max}/u_{\max}) \tag{29}$$

Using this result we obtain the following control law.

Control Law DGP 3:

$$\begin{aligned}
u&=u_{\max} && \text{if } u_i\ge u_{\max}\\
&=u_i && \text{if } |u_i|<u_{\max}\\
&=-u_{\max} && \text{if } u_i\le -u_{\max}
\end{aligned} \tag{30}$$

where

$$u_i=\frac{3M_p}{(1-v_{\max}/u_{\max})(t_f-t)^2}$$

IV. Guidance Laws: First-Order Missile Airframe/Autopilot Response

While the missile is modeled with a first-order response in this section, the target model still is based on a perfect response. This has especially important ramifications in the guidance laws based on differential game theory.

Optimal Control Approach

With the target control v considered as an assumed constant $\bar{v}$ for the remainder of the flight, the necessary optimality conditions to minimize Eq. (9) gives the TPBVP

$$\begin{aligned}
&\dot{M}_p=(\bar{v}-y_3)(t_f-t) && M_p(t_0)\ \text{known}\\
&\dot{y}_3=(u^*-y_3)/\tau && y_3(t_0)\ \text{known}\\
&\dot{\lambda}_m=0 && \lambda_m(t_f)=M_p(t_f)\\
&\dot{\lambda}_3=\lambda_m(t_f-t)+\lambda_3/\tau && \lambda_3(t_f)=0\\
&u^*=-u_{\max}\operatorname{sgn}\lambda_3 && \text{if } \lambda_3\neq 0\\
&u^*\quad \text{nonunique} && \text{if } \lambda_3=0
\end{aligned} \tag{31}$$

This TPBVP can be solved to obtain

$$u=u_{\max}$$

if

$$M_p>(u_{\max}-\bar{v})(t_f-t)^2/2 -\tau(y_3-u_{\max})[(t_f-t)-\tau(1-e^{-(t_f-t)/\tau})]$$

and

$$u=-u_{\max} \tag{32}$$

if

$$M_p<(-u_{\max}-\bar{v})(t_f-t)^2/2 -\tau(y_3+u_{\max})[(t_f-t)-\tau(1-e^{-(t_f-t)/\tau})]$$

Nonunique solutions exist when

$$M_p<(u_{\max}-\bar{v})(t_f-t)^2/2 -\tau(y_3-u_{\max})[(t_f-t)-\tau(1-e^{-(t_f-t)/\tau})] \tag{33}$$

and

$$M_p>(-u_{\max}-\bar{v})(t_f-t)^2/2 -\tau(y_3+u_{\max})[(t_f-t)-\tau(1-e^{-(t_f-t)/\tau})]$$

A control law analogous to OCP 2 can be specified for this formulation with a first-order missile response by noting that zero final miss will be achieved, based on the target's predicted control $\bar{v}$, with $u^*=0$ if the missile remains on the state defined by

$$y_3(t)=0 \qquad M_p+\bar{v}(t_f-t)^2/2=0 \tag{34}$$

This optimal control, first-order autopilot (OCF) law can be stated as follows.

Control Law OCF 1:

If

$$|y_3(t)| > y_{3\text{tol}} \quad \text{or} \quad |M_p + \bar{v}(t_f - t)^2/2| > M_{p\text{tol}}$$

use $u^* = \pm u_{\max}$ to drive the trajectory to the following state in minimum time:

$$y_3 = 0 \qquad M_p + \bar{v}(t_f - t)^2/2 = 0$$

If

$$|y_3(t)| \leq y_{3\text{tol}} \quad \text{and} \quad |M_p + \bar{v}(t_f - t)^2/2| \leq M_{p\text{tol}} \tag{35}$$

use $u^* = 0$ where $y_{3\text{tol}}$ is a tolerance on y_3. Complete details on this control law are given in Ref. 1.

An alternate approach to the definition of the nonunique control is to enforce $M_p(t_f) = 0$ while minimizing the control energy defined by Eq. (15). The resulting TPBVP is

$$\dot{M}_p = (\bar{v} - y_3)(t_f - t) \qquad M_p(t_0) \text{ known, } M_p(t_f) = 0$$

$$\dot{y}_3 = (u^* - y_3)/\tau \qquad y_3(t_0) \text{ known}$$

$$\dot{\lambda}_m = 0$$

$$\dot{\lambda}_3 = \lambda_m(t_f - t) + \lambda_3/\tau \qquad \lambda_3(t_f) = 0$$

$$\begin{aligned} u^* &= u_{\max} && \text{if} \quad -\lambda_3/\tau \geq u_{\max} \\ &= -\lambda_3/\tau && \text{if} \quad -u_{\max} < -\lambda_3/\tau < u_{\max} \\ &= -u_{\max} && \text{if} \quad -\lambda_3/\tau \leq -u_{\max} \end{aligned} \tag{36}$$

This TPBVP can be solved to obtain the following feedback control law.

Control Law OCF 2:

$$\begin{aligned} u^* &= u_{\max} && \text{if} \quad u_i \geq u_{\max} \\ &= u_i && \text{if} \quad |u_i| < u_{\max} \\ &= -u_{\max} && \text{if} \quad u_i \leq -u_{\max} \end{aligned} \tag{37}$$

where

$$u_i = \frac{[M_p + \bar{v}(t_f - t)^2/2 - y_3 F_1(t)] F_2(t)}{(t_f - t)^3/3 - F_3(t) F_1(t) + F_4(t)}$$

$$F_1(t) = \tau(t_f - t) - \tau^2(1 - e^{-(t_f - t)/\tau})$$

$$F_2(t) = (t_f - t) - \tau(1 - e^{-(t_f - t)/\tau})$$

$$F_3(t) = t_f - t + (\tau/2) e^{-(t_f - t)/\tau}$$

$$F_4(t) = (\tau^3/2)\left\{1 - \left[\frac{(t_f - t)}{\tau} + 1\right] e^{-(t_f - t)/\tau}\right\}$$

Differential Game Approach

When the missile response is modeled as a first-order lag, the target is always able to force a nonzero value of $M_p(t_f)$ if $v_{\max} > 0$ regardless of the value of $u_{\max}$. The reason for this is that this lag prevents the missile from being able to react sufficiently fast to maneuvers by the target just prior to intercept.

For a saddle point of the performance index given by Eq. (9), the TPBVP is

$$\dot{M}_p = (v^* - y_3)(t_f - t) \qquad M_p(t_0) \text{ known}$$

$$\dot{y}_3 = (u^* - y_3)/\tau \qquad y_3(t_0) \text{ known}$$

$$\dot{\lambda}_m = 0 \qquad \lambda_m(t_f) = M_p(t_f)$$

$$\dot{\lambda}_3 = \lambda_m(t_f - t) + \lambda_3/\tau \qquad \lambda_3(t_f) = 0$$

$$u^* = -u_{\max}\,\text{sgn}\lambda_3 \text{ and } v^* = -v_{\max}\,\text{sgn}\lambda_3 \text{ if } \lambda_3 \neq 0$$

$$u^*, v^* \text{ undefined if } \lambda_3 = 0 \tag{38}$$

Integration of this TPBVP shows that for values of t sufficiently near t_f, the controls can be expressed as

$$u^* = u_{\max}\,\text{sgn}M_p(t_f) \qquad v^* = v_{\max}\,\text{sgn}M_p(t_f) \tag{39}$$

i.e., in this region the saddle point controls are constant with the same sign.

Integration of the TPBVP with $u = u_{\max}$, $v = v_{\max}$, and then with $u = -u_{\max}$, $v = -v_{\max}$, gives the following surface from which the value of $|M_p(t_f)]$ obtained is the same regardless of the signs of u^* and v^*.

$$M_p - \tau^2 y_3 [e^{-(t_f - t)/\tau} - 1 + (t_f - t)] = 0 \tag{40}$$

By keeping the state on this surface as long as possible, the missile is able to minimize the maximum value of $|M_p(t_f)|$ which can be forced by the target. The target must force the state off of this surface by playing $v = v_{\max}$ or $v = -v_{\max}$.

The optimal time t^* for the evader to initiate this maneuver to maximize $|M_p(t_f)|$ is found from the transcendental equation

$$\left(\frac{v_{\max}}{u_{\max}} - 1\right)\frac{(t_f - t^*)}{\tau} - e^{-(t_f - t^*)/\tau} + 1 = 0 \tag{41}$$

For $t < t^*$, the missile is able to keep the state on the surface defined by Eq. (40) by using the control

$$u = \frac{\hat{v}(t_f - t)/\tau}{e^{-(t_f - t)/\tau} - 1 + (t_f - t)/\tau} \tag{42}$$

where $\hat{v}$ is the estimated current value of target acceleration. These results can be summarized by the following missile differential game, first-order autopilot (DGF) control law which is designed to maintain the system state on the surface defined by Eq. (40) as long as possible.

Control Law DGF 1:

$$\begin{aligned} u^* &= u_{\max} && \text{if} \quad \tilde{M}_p > M_{p\text{tol}} \\ &= -u_{\max} && \text{if} \quad \tilde{M}_p < M_{p\text{tol}} \\ &= \frac{\hat{v}(t_f - t)/\tau}{e^{-(t_f - t)/\tau} - 1 + (t_f - t)/\tau} && \text{if} \quad |\tilde{M}_p| \leq M_{p\text{tol}} \end{aligned} \tag{43}$$

where

$$\tilde{M}_p = M_p - \tau^2 y_3 [e^{-(t_f - t)\tau} - 1 + (t_f - t)/\tau]$$

V. Simulation Results

Preliminary testing of the guidance laws was accomplished using the TACTICS III simulation program that employs a point mass missile model with realistic aerodynamics. For these preliminary evaluations, perfect seeker dynamics and a first-order airframe/autopilot response with time constant

$\tau = 0.4$ s are assumed. The missile thrust profile consists of a maximum thrust of 4711 lb for 2.6 s, followed by a coast. The launch weight is 165 lb and the burnout weight is 115 lb. The planar guidance laws from the previous two sections were implemented in both vertical and local horizontal channels.

Two different scenarios were used. The first is a close-in beam attack with launch conditions shown in Fig. 2. Throughout the flight, the target executes a 9 g turn into the missile. This scenario requires that the missile employ maximum lateral control for most of the flight to achieve a near miss or a hit. For this scenario, the flight time is less than 2 s and the maximum allowable control is determined by the C_L limit of the missile rather than the structural limit which was set at 100 g's.

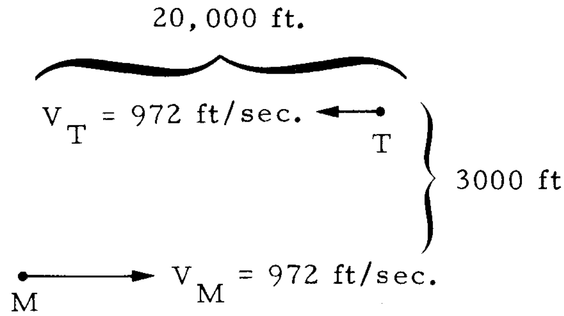

Fig. 2 Initial conditions for scenario 1.

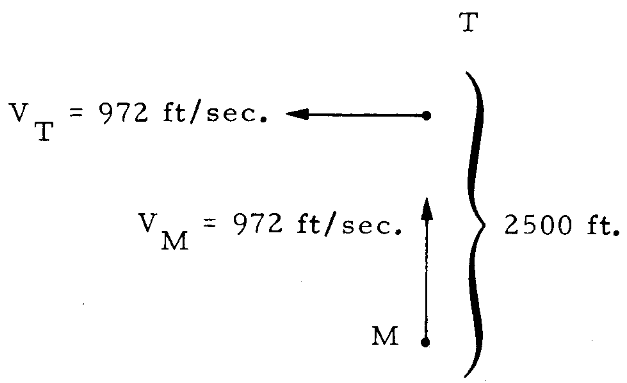

Fig. 3 Initial conditions for scenario 2.

The launch conditions of scenario 2 are shown in Fig. 3. This is a long-range intercept problem in which the structural limit of the missile is reduced to 30 g's so that the target can out-turn the missile just prior to intercept. The target flies straight and level from the time of launch to about 4.5 s after launch, and then executes a 9 g turn toward the missile.

The control laws were tested with quantity $\bar{v}$ set to the actual current target acceleration component normal to the LOS and then with $\bar{v} = 0$, i.e., no knowledge of target acceleration is assumed. Note that control laws DGP 2 and DGP 3 do not depend on $\bar{v}$.

The miss distances produced by the guidance laws with these two scenarios and the two assumptions on $\bar{v}$ are summarized in Table 1.

One major conclusion from the results in Table 1 is that the differential game guidance laws are much less sensitive to errors in estimates of target acceleration than are the optimal control guidance laws. This is especially true for scenario 2 and results from the fact that the differential game guidance philosophy is based on the maneuver capabilities of the target and not a specific anticipated target maneuver.

The guidance laws based on a perfect missile airframe/autopilot response perform better for scenario 2 than do the first-order laws. Conversely, the first-order laws are better for the highly dynamic scenario 1 that is near the inner launch boundary. Note that the miss distances attained with OCP 1/DGP 1 remain inside or very close to M_{ptol} for scenario 2, but appear to reach a lower limit of between 6 and 7 ft for scenario 1 as M_{ptol} is reduced. The corresponding first-order law DGF 1 remains within M_{ptol} for scenario 1, but exceeds this quantity in all cases for scenario 2. The reason for this poor relative performance of DGF 1 compared to DGP 1 in scenario 2 is that more corrections with $u^* = \pm u_{max}$ are required for DGF 1. These maximum turn rate arcs apparently bleed off sufficient energy so that the missile lacks sufficient maneuverability near intercept to hit the target.

Table 1 Miss distances (ft) from guidance law simulations

	Scenario 1		Scenario 2	
	$\bar{v}=0$	$\bar{v}$(true)	$\bar{v}=0$	$\bar{v}$(true)
OCP 1/DGP 1				
$M_{ptol}=10$	7.8	8.4	9.2	9.0
$M_{ptol}=5$	6.1	6.2	5.1	...
$M_{ptol}=2$	6.3	6.3	2.1	...
OCP 2				
$M_{ptol}=10$	21.1	21.1	9.6	7.6
OCP 3	14.2	16.5	45.1	1.5
DGP 2	7.2		16.0	
DGP 3	16.1		0	
OCF 1				
$M_{ptol}=10$	18.2	18.2	61.7	14.5
OCF 2	1.5	4.8	61.5	2.5
DGF 1				
$M_{ptol}=10$	8.3	1.2	25.2	16.9
$M_{ptol}=5$	4.0	...	19.8	15.6
$M_{ptol}=2$	1.4	...	19.8	19.2

VI. Conclusions

The structure of the optimal control and differential game laws is usually different for similar dynamics and performance indices. In the laws based on minimization of control energy (OCP 3, DGP 2, and DGP 3), the gains in the differential game guidance laws are functions of the maneuver capability of the target relative to the missile, whereas no such dependence exists for the optimal control laws. In addition, differential game formulations in which zero terminal miss is specified are prohibited in the case of a first-order missile airframe/autopilot response and a perfect target response, while a specified zero terminal miss optimal control formulation is always allowed since the law is based on an assumed future target control profile.

The major advantage of differential game guidance laws compared to similar laws based on optimal control theory is that the differential game laws are less sensitive to errors in estimates of target acceleration. This results from the fact that the differential game guidance laws are based only on the maneuver capabilities of the target and not a projected future acceleration history as is required for the optimal control laws. Therefore, the use of differential game methods in the design of guidance laws for intercept missiles appears to result in better missile performance against highly maneuverable targets than does the use of optimal control theory.

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